# Contents

1. Introduction to The Weil conjectures and \( l \)-adic monodromy 2  
   1.1. A history of the proof of the Weil conjectures 2  
   1.2. \( p \)-adic base fields and \( l \)-adic monodromy 2  
   1.3. Towards weight filtration 3  
   1.4. Sheaves of nearby cycles 3  
2. Sheaf Theory 3  
3. Etale Cohomology 3  
4. Cohomology of Curves, Poincare Duality and Artin Comparison 3  
5. Nearby Cycles Sheaves 3  
6. Application to Local-Global \( GL_2/\mathbb{Q} \) 3  
7. Computing \( R\psi\bar{\mathbb{Q}}_l \) in the Semistable Case 3  
8. Construction of the Weight Spectral Sequence 3  
9. Application of T. Saito’s Proof of \( l \)-Independence 3  
References 3
1. Introduction to the Weil conjectures and \( l \)-adic monodromy

1.1. A history of the proof of the Weil conjectures. Throughout this section, \( X \) will denote a scheme of finite type over a finite field \( k = \mathbb{F}_q \).

**Definition 1.** Denote \( k_n = \mathbb{F}_{q^n} \) and \( Z(u) \) such that \( u \frac{d}{du} \log Z(u) = \sum_{n=1}^{\infty} |X(k_n)|u^n \).

A sheaf \( \mathcal{F} \) on \( \bar{\mathbb{Q}}_l \) is called
- (pointwise) **pure of weight** \( i \) if for every closed point \( x \in |X| \), the eigenvalue of the geometric Frobenius map \( \text{Frob}_x \in \text{Gal} \left( \mathbb{F}_q(x)/k(x) \right) \) on \( \mathcal{F}|_x \) is an algebraic integer all of whose complex conjugates have absolute value \( |k(x)|^{i/2} \).
- mixed if it is a successive extension of pure sheaves.

**Theorem 2.** Weil-Deligne: Let \( X \) be a proper smooth variety over \( k = \mathbb{F}_q \) that is pure of dimension \( d \).

1. \( Z(u) = \frac{P_1(u)P_2(u)\ldots P_{2d-1}(u)}{P_0(u)P_2(u)\ldots P_{2d-2}(u)} \) with each \( P_i(u) \in \mathbb{Z}[u] \)
2. **Functional equation**
3. **Riemann hypothesis:** all roots of \( P_i(u) \) have absolute value \( q^{-i/2} \)

**Proof.** (History)
- Dwork proved rationality (1) using cohomology.
- Grothendieck invented \( l \)-adic cohomology to obtain proofs of (1) and (2); a standard conjecture implies (3).
- Grothendieck’s idea and Berthelot’s proof of the “char(\( k \))–adic” i.e. crystalline cohomology which also gives (1) and (2).
- Deligne ([Weil II]) proved (3) for \( X/k \) projective and smooth by using Lefschetz pencils and monodromy.
- Deligne (in [Weil III]) introduces the “correct” structure on Weil cohomology, this implies (3): the ideas of weight, purity and mixedness give a proof of (3) without the projective requirement.

**Example 3.** \( \overline{\mathbb{Q}}_l \) is pure of weight 0. \( \overline{\mathbb{Q}}_l(1) := \left( \lim_{n \to \infty} \mu_n \right) \otimes \mathbb{Z}_l \) is pure of weight = 2.

**Theorem 4.** (due to Deligne) If \( X/k \) is proper and smooth and \( \mathcal{F} \) lisse \( \overline{\mathbb{Q}}_l \)-sheaf on \( X \) pure of weight \( \beta \) then \( H^i(X, \mathcal{F}) \) is pure of weight \( \beta + 1 \).

(Further, if \( X/k \) is of finite type then \( H^i(X, \mathcal{F}) \) is mixed with weight \( \beta + 1 \)).

In fact, Deligne proves that \( R^i f_! \) preserves mixedness for \( X \to Y \).

1.2. \( p \)-adic base fields and \( l \)-adic monodromy. Note that if \( k \) is any field and \( X/k \) proper then \( H^i(X, \overline{\mathbb{Q}}_l) \) is a finite dimensional \( \overline{\mathbb{Q}}_l \)-vector space. It has a continuous group action of \( \text{Gal}(\bar{k}/k) \) upon it. For the rest of this section, assume \( K : \mathbb{Q}_p \) is a finite extension with ring of integers \( \mathcal{O}_K \) and residue field \( k \) of characteristic \( p > 0 \).

**Remark 5.** Given \( X/K \) proper and smooth, how mixed is \( H^i(X, \overline{\mathbb{Q}}_l) \)?

**Lemma 6.** (Tate exact sequence for the inertia group) Let \( I_K \) be the inertia group then there is a short exact sequence \( 1 \to P_{K,l} \to I_K \xrightarrow{l} \mathbb{Z}_l(1) \to 0 \).

**Theorem 7.** (\( l \)-adic monodromy)
Let \( (V, \rho) \) be an \( l \)-adic \( \text{Gal}(\bar{K}/K) \)-representation then \( \rho|_{I_K} \) is quasi-unipotent.

Denoting \( V(1) := V \otimes_{\mathbb{Q}_p} \) then the above says that \( \exists \text{ nilpotent } N : V(1) \to V \) that is \( \text{Gal}(\bar{K}/K) \)-equivariant such that \( \exists N' \subset I_K \) with \( \rho|_{N'} \) unipotent and for any such \( N' \) and \( \forall g \in I'; \rho(g) = \exp(Nt_l(g)) \).

In this section assume \( (\rho, V) \) and \( N \) as in the \( l \)-adic monodromy theorem.

**Definition 8.** A filtration is called exhaustive if all sufficiently large terms are the original object. It is separated if all sufficiently small terms are 0.
Theorem 9. (The monodromy filtration)

\[ \exists! \text{ increasing separated exhaustive filtration } M_i V \text{ such that} \]

1. \( N(M_i V)(1) \subset M_{i-2} V \)

2. \( N^m \text{ induces } gr^M_m (m) \to gr^M_m (V) \)

Conjecture 10. (Weight-Monodromy) \( V = H^i(X_{\overline{\mathbb{Q}}_l}, \mathbb{Q}_l) \) with \( X/K \) proper, smooth.

Then \( gr^M_m V \) is pure of weight \( i + m \) and so \( M_i V = V, M_{i-1} V = 0 \).

Proof. The result is currently known in the following cases:

- (Deligne, Ito and Terasuma) when \( \text{char } K > 0 \).
- (Zink, Rapoport) when \( K/\mathbb{Q}_p \) finite with \( \text{dim } X \leq 2 \).

1.3. Towards weight filtration.

1.4. Sheaves of nearby cycles.

2. Sheaf Theory

3. Étale Cohomology

4. Cohomology of Curves, Poincare Duality and Artin Comparison

5. Nearby Cycles Sheaves

6. Application to Local-Global \( GL_2/\mathbb{Q} \)

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References